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Co-efficient problem for a new subclass of analytic functions using subordination

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Abstract

In this paper, by taking the analytic function $\phi(z)$, *we obtained the sharp upper bounds of the Fekete-Szego functional* $\left|a_3 - \mu a_2^2\right|$ *and defined the general class* $M_{g,h}^{\alpha,\delta}(\phi)$ *by using the convolution and subordinate. We also discussed some applications of our main result.*

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1 INTRODUCTION

Let A be the class of analytic function f which is defined on the unit disk $\Delta := \{z \in C : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

Let the class A has the subclass S which consists the univalent functions. For two analytic functions f and g, the function f is subordinate to g in the unit disk Δ denoted by $f \prec g$, for the analytic function w with $|w(z)| \le |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

A function $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is said to be in the class P if $\operatorname{Re} p(z) > 0$. In the open unit disk Δ , the analytic univalent function ϕ has positive real part and $\phi(\Delta)$ be symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. Ma and Minda [4] introduced the classes $S^*(\phi)$ and $C(\phi)$ satisfying $z f'(z)/f(z) \prec \phi(z)$ and $1 + z f''(z)/f'(z) \prec \phi(z)$ respectively, by presenting various subclasses of starlike and convex functions.

The class $M(\alpha, \phi)$ of $\alpha-$convex functions with respect to ϕ which consists the function f in A and includes several classes as $S^*(\phi)$, $C(\phi)$ and

$$M(\alpha, (1 + (1 - 2\alpha)z)/(1 - z)) =: M(\alpha),$$

Introduced by Ali et al [1] Which satisfying

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \phi(z).$$

Ali et al. [2] also introduced several coefficient problems for $p-$valent analytic functions. Miller and Mocanu [5] were introduced and studied the $\alpha-$convex function's class $M(\alpha)$.

In 1933 Fekete and Szego proved that

$$\left|a_2^2-\mu a_3\right|\le\begin{cases}4\mu-3 & (\mu\ge 1)\\ 1+\exp\left(\dfrac{-2\mu}{1-\mu}\right) & (0\le\mu\le 1)\\ 3-4\mu & (\mu\le 0)\end{cases}$$

By Fekete-Szego problem, we can find the sharp bounds for the non-linear functional $\left|a_3-\mu a_2^2\right|$ of any compact family of functions. In 1969, the sharp bounds for Fekete-Szego functional $\left|a_2^2-\mu a_3\right|$ for functions in some subclasses of S were obtained by Keogh and Merk [3].

We have the functions

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \qquad (1.1)$$

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n \qquad (1.2)$$

$$h(z)=z+\sum_{n=2}^{\infty}h_nz^n \qquad (1.3)$$

And the convolution of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z):=z+\sum_{n=2}^{\infty}a_ng_nz^n=:(g*f)(z).$$

Murugusundaramoorthy et al. [6] introduced a new class $M_{g,h}$ of functions $f\in A$, by using the Hadamard product, which satisfying

$$\frac{(f*g)(z)}{(f*h)(z)}\prec\phi(z)$$

Where $g,h\in A,(g_n>0,h_n>0,g_n-h_n>0)$

Our results extend several earlier known works in [3, 4, 6].

Definition 1.1 Let the functions $g(z)$ and $h(z)$ defined in (1.2) and (1.3) respectively with $g_n>0,\quad h_n>0$ and $g_n-h_n>0$ for the analytic function ϕ with $\phi(0)=1$ and $\phi'(0)>0,$ the function $f\in A$ given by (1.1) is said to be in the class $M_{g,h}^{\alpha,\delta}(\phi)$, for $\alpha>0$, if it satisfies

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)}+\alpha\frac{(f*g)'(z)}{(f*h)'(z)}\prec\phi(z)$$

To prove our main result, we required the following Lemma 1.1 of Ali et al. [2]. Let Ω be the class of analytic functions w, with conditions $w(0)=0,|w(z)|<1.$

1.1 Lemma

If $w\in\Omega$ and $w(z)=w_1z+w_2z^2+\cdots(z\in\Delta)$, then

$$\left|w_2-tw_1^2\right|\le\begin{cases}-t & (t\le -1)\\ 1 & (-1\le t\le 1)\\ t & (t\ge 1)\end{cases}$$

For $t<-1$ or $t>1$, equality holds if and only if $w(z)=z$ or one of its rotations. For $-1<t<1$, equality holds if and only if $w(z)=z^2$ or one of its rotations. Equality holds for $t=-1$ if and only if

$$w(z)=z(\lambda+z)/(1+\lambda z) \qquad (0\le\lambda\le 1)$$

Or one of its rotations, while for $t=1$, equality holds if and only if

$$w(z)=-z(\lambda+z)/(1+\lambda z) \qquad (0\le\lambda\le 1)$$

or one of its rotations.

1.2 Lemma

If $w\in\Omega$, then, for any complex number t,

$$\left|w_2-tw_1^2\right|\le\max\{1;|t|\}$$

And the result is sharp for the functions given by $w(z)=z^2$ or $w(z)=z.$

2. FEKETE-SZEGO PEOBLEM

We begin with the following result:

Theorem 2.1

Let $\phi(z)=1+B_1z+B_2z^2+\cdots$. If $f(z)$ given by (1.1) belongs to the class $M_{g,h}^{\alpha,\delta}(\phi)$, then, for any real number μ,

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{B_1A\delta}{(1+2\alpha)(g_3-h_3)} & (\mu\le\sigma_1)\\ \dfrac{B_1\delta}{(1+2\alpha)(g_3-h_3)} & (\sigma_1\le\mu\le\sigma_2)\\ \dfrac{B_1A\delta}{(1+2\alpha)(h_3-g_3)} & (\mu\ge\sigma_2)\end{cases}$$

Where

$$A=\frac{B_2}{B_1}+\frac{(\delta-1)}{2}B_1-\frac{\delta\left[(1+3\alpha)(h_2^2-h_2g_2)+\mu(1+2\alpha)(g_3-h_3)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2}$$

$$\sigma_1=\frac{(\delta-1)(1+\alpha)^2(g_2-h_2)^2}{2(1+2\alpha)(g_3-h_3)\delta}+\frac{(B_2-B_1)(1+\alpha)^2(g_2-h_2)^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)}-\frac{\delta(1+3\alpha)(h_2^2-h_2g_2)B_1^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)}$$

And

$$\sigma_2=\frac{(\delta-1)(1+\alpha)^2(g_2-h_2)^2}{2(1+2\alpha)(g_3-h_3)\delta}+\frac{(B_2+B_1)(1+\alpha)^2(g_2-h_2)^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)}-\frac{\delta(1+3\alpha)(h_2^2-h_2g_2)B_1^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)}$$

And for any complex number μ

$$\left|a_3-\mu a_2^2\right|\le\frac{B_1\delta}{(1+2\alpha)(g_3-h_3)}\max\{1;|t|\}$$

Where

$$t=\frac{\delta\left[(1+3\alpha)(h_2^2-h_2g_2)+\mu(1+2\alpha)(g_3-h_3)\right]B_1^2}{(1+\alpha)^2(g_2-h_2)^2B_1}-\frac{B_2(1+\alpha)^2(g_2-h_2)^2}{(1+\alpha)^2(g_2-h_2)^2B_1}-\frac{(\delta-1)}{2}B_1$$

Proof . If $f\in M_{g,h}^{\alpha,\delta}(\phi)$, then there exist an analytic function

$$w(z)=w_1z+w_2z^2+\cdots\in\Omega$$

Such that

$$(1-\alpha)\frac{(f*g)(z)}{(f*h)(z)}+\alpha\frac{(f*g)'(z)}{(f*h)'(z)}=\phi(w(z))^\delta \quad (2.3)$$

By using

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

$$g(z)=z+\sum_{n=2}^{\infty}g_nz^n \text{ and } h(z)=z+\sum_{n=2}^{\infty}h_nz^n$$

We get

$$\frac{(1-\alpha)\left(z+\sum_{n=2}^{\infty}a_ng_nz^n\right)}{z+\sum_{n=2}^{\infty}a_nh_nz^n}+\frac{\alpha\left(z+\sum_{n=2}^{\infty}a_ng_nz^n\right)'}{\left(z+\sum_{n=2}^{\infty}a_nh_nz^n\right)'}$$

Then the computation shows that

$$\frac{(f*g)(z)}{(f*h)(z)} = 1 + a_2(g_2 - h_2)z + [a_3(g_3 - h_3)]z^2 + [a_2^2(h_2^2 - h_2g_2)]z^2 + \cdots \quad (2.4)$$

$$\frac{(f*g)'(z)}{(f*h)'(z)} = 1 + a_2(g_2 - h_2)z + [3a_3(g_3 - h_3)]z^2 + [4a_2^2(h_2^2 - h_2g_2)]z^2 + \cdots \quad (2.5)$$

And

$$(\phi(w(z)))^{\delta} = 1 + \delta\left(B_1w_1z + \left(B_1w_2 + B_2w_1^2\right)z^2\right) + \frac{\delta(\delta-1)}{2}\left(B_1^2w_1^2z^2\cdots\right) \quad (2.6)$$

From (2.3), (2.4), (2.5) and (2.6), we get

$$(1+\alpha)(g_2 - h_2)a_2 = \delta B_1 w_1$$

$$a_2 = \frac{\delta B_1 w_1}{(1+\alpha)(g_2 - h_2)} \quad (2.7)$$

And

$$(1+2\alpha)(g_3 - h_3)a_3 + (1+3\alpha)(h_2^2 - h_2g_2)a_2^2 = \delta\left(B_1w_2 + B_2w_1^2\right) + \frac{\delta(\delta-1)}{2}B_1^2w_1^2$$

$$a_3 = \frac{1}{(1+2\alpha)(g_3 - h_3)}\Bigg[\delta B_1w_2 + \delta B_2w_1^2 + \frac{\delta(\delta-1)}{2}B_1^2w_1^2 - (1+3\alpha)(h_2^2 - h_2g_2)\frac{\delta B_1^2w_1^2}{(1+\alpha)^2(g_2-h_2)^2}\Bigg] \quad (2.8)$$

A computation using (2.7) and (2.8) gives

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1\delta}{(1+2\alpha)(g_3 - h_3)}\left[w_2 - tw_1^2\right] \quad (2.9)$$

Where

$$t = -\frac{B_2}{B_1} + \frac{\delta\left[(1+3\alpha)(h_2^2 - h_2g_2) + \mu(1+2\alpha)(g_3 - h_3)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} - \frac{(\delta-1)}{2}B_1 \quad (2.10)$$

Now the first inequality (1.3) is established as follows by an application of Lemma 1.1.If

$$-\frac{B_2}{B_1} + \frac{\delta\left[(1+3\alpha)(h_2^2 - h_2g_2) + \mu(1+2\alpha)(g_3 - h_3)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} - \frac{(\delta-1)}{2}B_1 \le -1,$$

Then

$$\mu_1 \le \frac{(\delta-1)(1+\alpha)^2(g_2-h_2)^2}{2(1+2\alpha)(g_3-h_3)\delta} + \frac{(B_2 - B_1)(1+\alpha)^2(g_2-h_2)^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)} - \frac{\delta(1+3\alpha)(h_2^2 - h_2g_2)B_1^2}{\delta B_1^2(1+2\alpha)(g_3-h_3)} = \sigma_1$$

And Lemma 1.1 gives

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1 A\delta}{(1+2\alpha)(g_3 - h_3)}$$

For

$$-1 \le -\frac{B_2}{B_1} + \frac{\delta\left[(1+3\alpha)(h_2^2 - h_2g_2)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} + \frac{\delta\left[\mu(1+2\alpha)(g_3 - h_3)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} - \frac{(\delta-1)}{2}B_1 \le 1,$$

We have $\sigma_1 \le \mu \le \sigma_2$, where σ_1 and σ_2 are as given in the statement of the theorem.

Now an application of Lemma 1.1 yields

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1\delta}{(1+2\alpha)(g_3-h_3)}$$

For

$$-\frac{B_2}{B_1} + \frac{\delta\left[(1+3\alpha)\left(h_2^2 - h_2 g_2\right)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} + \frac{\delta\left[\mu(1+2\alpha)(g_3-h_3)\right]B_1}{(1+\alpha)^2(g_2-h_2)^2} - \frac{(\delta-1)}{2}B_1 \ge 1,$$

We have $\mu \ge \sigma_2$ and it follows from Lemma 1.1 that

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1 A\delta}{(1+2\alpha)(g_3-h_3)}$$

Now the second inequality (2.2) follows by an application of lemma 1.2 as follows:

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1\delta}{(1+2\alpha)(g_3-h_3)}\left[w_2 - t w_1^2\right] \le \frac{B_1\delta}{(1+2\alpha)(g_3-h_3)}\max\{1:|t|\},$$

Where t is given by (2.10).

The extremal functions for the first and third inequality of our main result $M_{g,h}^{\alpha,\delta}(\phi)$ is

$$f_1(z) = z(1+nz)^m$$

Where $n = \frac{b_1^2 - 2a_1}{b_1}$ and $m = \frac{b_1^2}{b_1^2 - 2a_1}$,

$$a_1 = \frac{B_2}{B_1} + \frac{(\delta-1)}{2}B_1 - \frac{\delta(1+3\alpha)\left(h_2^2 - g_2 h_2\right)}{(1+\alpha)^2(g_2-h_2)^2}$$

and $b_1 = \frac{\mu\delta B_1(1+2\alpha)(g_3-h_3)}{(1+\alpha)^2(g_2-h_2)^2}$

The extremal function for the second inequality is

$$f_2(z) = z\left(1+z^2\right)^{\frac{B_1\delta}{(1+2\alpha)(g_3-h_3)}}$$

Remark

If $\delta = 1$ in our main result $M_{g,h}^{\alpha,\delta}(\phi)$,

$$M_{g,h}^{\alpha,1}(\phi) = M_{g,h}^{\alpha}(\phi)$$

Then we obtain

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1 A}{(1+2\alpha)(g_3-h_3)} & (\mu \le \sigma_1) \\ \frac{B_1}{(1+2\alpha)(g_3-h_3)} & (\sigma_1 \le \mu \le \sigma_2) \\ \frac{B_1 A}{(1+2\alpha)(h_3-g_3)} & (\mu \ge \sigma_2) \end{cases}$$

This result is obtained by [7]

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